

Bulk Viscous Matter-dominated Universes: Asymptotic Properties

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By means of a combined study of the type Ia supernovae test, together with a study of the asymptotic properties in the equivalent phase space – through the use of the dynamical systems tools – we demonstrate that the bulk viscous matter-dominated scenario is not a good model to explain the accepted cosmological paradigm, at least, under the parametrization of bulk viscosity considered in this paper. The main objection against such scenarios is the absence of conventional radiation and matter-dominated critical points in the phase space of the model. This entails that radiation and matter dominance are not generic solutions of the cosmological equations, so that these stages can be implemented only by means of very particular solutions. Such a behavior is in marked contradiction with the accepted cosmological paradigm which requires of an earlier stage dominated by relativistic species, followed by a period of conventional non-relativistic matter domination, during which the cosmic structure we see was formed.

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I. INTRODUCTION

Within the context of early inflation, it has been known since long time ago that an imperfect fluid with bulk viscosity in cosmology can produce an accelerated expansion without the need of a cosmological constant or some other inflationary scalar field [1] (although some authors do not agree with this conclusion [2]). Further extrapolation of this idea – used to induce an accelerated expanding Universe without the need of unknown components – leads to the possibility that one alternative candidate to explain the present acceleration can be bulk viscous pressure [3]. No matter how attractive it seems, this idea faces some problems, among them, the need to have a satisfactory mechanism for the origin and composition of the bulk viscosity (see, for instance [4, 5]).

From a thermodynamical point of view the bulk viscosity in a physical system is due to its deviations from the local thermodynamic equilibrium [5]. In a cosmological setting, the bulk viscosity may arise when the cosmic fluid expands (or contracts) too fast so that the system does not have enough time to restore its local thermodynamic equilibrium and then it arises an effective pressure restoring the system to its thermal equilibrium. When the fluid reaches again the thermal equilibrium then the

bulk viscous pressure vanishes [5, 6]. Therefore, in an accelerated expanding Universe, it may be natural to assume the possibility that the expansion process is actually a collection of states out of thermal equilibrium in a small fraction of time giving rise to the existence of a bulk viscosity [7].

Usually the way to test the theoretical (and observational) viability of a given cosmological model is through using known solutions of the cosmological field equations, or by seeking for new particular solutions that are physically plausible. However, in general, the cosmological field equations are very difficult to solve exactly. Even if a given analytic solution can be found, it will not be unique, but just one in a large set of them. This is not to talk about stability of given solution(s). An alternative way around is to invoke the dynamical systems tools to extract very useful information about the asymptotic properties of the model instead. In this regard, knowledge of the critical (also equilibrium or singular) points in the phase space – corresponding to a given cosmological model – is a very important information since, independent on the initial conditions chosen, the orbits of the corresponding autonomous system of ordinary differential equations (ODE) will always evolve for some time in the neighborhood of these points. Besides, if the point were an (global) attractor, independent of the initial conditions, the orbits will always be attracted towards it (either into the past or into the future). Going back to the original cosmological model, the existence of the equilibrium points can be correlated with generic cosmological solutions that might really decide the fate and/or the origin of the cosmic evolution.

A phase space for a model which is consistent with the

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presently accepted cosmological paradigm should contain critical points associated with: i) a radiation-dominated (relativistic) stage, followed by ii) a matter-dominated (non-relativistic) phase, which is important to allow for the formation of the amount of cosmic structure we see, and iii) an accelerated expanding stage which possibly might last for ever. Perhaps, there should be also a critical point in the phase space associated with an early inflationary period, however, this would require of further refinements of a given cosmological model which is primarily intended to explain the period lasting between decoupling of radiation and baryons and up to the present accelerating phase.

In the present paper we will make a combined use of the dynamical systems tools and of the type Ia supernovae test, to extract as much as possible useful information about the asymptotic properties of a bulk viscous matter-dominated Universe, in order to be able to judge about its theoretical viability to accommodate the accepted cosmological paradigm. In the specific model we shall be investigating the cosmological dynamics is fuelled by a conventional matter component jointly with a fluid with bulk viscosity of the form: $\zeta = \zeta_0 + \zeta_1 H + \zeta_2 \ddot{a}/\dot{a}$, where ζ_0 , ζ_1 , and ζ_2 , are constants to be determined by the observations and $H = \dot{a}/a$ is the Hubble parameter. The term ζ_0 takes into account the simplest parametrization for the bulk viscosity: a constant. The term $\zeta_1 H$ characterizes the possibility of a bulk viscosity proportional to the expansion ratio of the Universe, while the third term $\zeta_2 \ddot{a}/\dot{a}$ takes into consideration the influence acceleration of the expansion might has on the bulk viscosity.

The results of the present study will convincingly show that the model of bulk viscosity considered here (see also [7, 8]) is in marked contradiction with the presently accepted cosmological paradigm and, hence, should be ruled out. In fact, it will be shown that there are not any critical points in the phase space of this cosmological model which could be associated with either radiation or matter domination. This, in turn, entails that radiation and matter dominance are not generic solutions of the model. These can be, at most, very particular solutions that can be achieved under specific initial conditions. This is in marked contradiction with the accepted cosmological paradigm in that the latter requires of an earlier stage dominated by relativistic species (specifically a radiation-dominated era), followed by a period of conventional matter domination during which the cosmic structure we see was formed.

The paper has been organized as it follows. The relevant details of the model are exposed in section II, followed by its observational testing using the supernovae SNe Ia data in section III A. In the first part of the dynamical systems study (section IV), for sake of simplicity, we will resort to a single component to embrace radiation, baryons, etc. Then, in section V, we consider a more physically involved model where the cosmic dynamics is fuelled by radiation, dark matter, and pressureless bulk viscous matter. In order to draw the phase portraits we

use the best estimated values of the free parameters of section III A. A thorough discussion of the properties of the model in the equivalent phase space is given in section VI. It will be evident that, independent of the values of the free parameters, there are not critical points in the phase space that could be correlated with either radiation or matter dominance stages. In the final section VII brief conclusions are given.

II. COSMOLOGY OF BULK VISCOUS MATTER-DOMINATED UNIVERSES.

In this section we analyze a cosmological model composed by a baryon matter and a bulk viscous components with dust behavior ($w = 0$) as dark matter, in a spatially flat universe. The energy-momentum tensor for the pressureless baryon component is given as a perfect fluid as usual

$$T_{\mu\nu}^{(m)} = \rho_m u_\mu u_\nu,$$

where ρ_m is the energy density of the matter component and u^μ is the four-velocity vector. The energy-momentum tensor of the bulk viscous component is that of an *imperfect fluid* with a first-order deviation from the thermodynamic equilibrium. It can be expressed as [9]:

$$T_{\mu\nu}^{(v)} = \rho_v u_\mu u_\nu + (g_{\mu\nu} + u_\mu u_\nu) P_v^*,$$

where

$$P_v^* \equiv P_v - \zeta \nabla_\nu u^\nu, \quad (1)$$

ρ_v and P_v are the energy density and pressure of the viscous fluid respectively. The term $g_{\mu\nu}$ is the metric tensor, and the subscript “v” stands for “viscous” component. The term ζ is a *bulk viscous* coefficient that arises in a fluid when it is out of the local thermodynamic equilibrium and which induces a *viscous pressure* $-\zeta \nabla_\nu u^\nu$ [5]. The term P_v^* is an effective pressure composed by the pressure P_v of fluid plus the bulk viscous pressure. It was initially proposed in reference [10] for relativistic dissipative processes in thermodynamic systems out of local equilibrium, and later on the authors of Ref. [11] developed an equivalent formulation.

Here we shall explore the cosmological implications of such a model by assuming that the Universe is filled with bulk viscous fluid only. For Friedmann-Robertson-Walker (FRW) spacetimes with flat spatial sections we have

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2),$$

where $a(t)$ is the scale factor. The conservation equations for the pressureless baryon matter and the viscous component can be written respectively as

$$\begin{aligned} \dot{\rho}_m + 3H\rho_m &= 0, \\ \dot{\rho}_v + 3H(\rho_v + P_v^*) &= 0, \end{aligned} \quad (2)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and, as usual, the over-dot stands for time derivative. The solution of the first equation in (2) for the matter component is $\rho_m = \rho_{m0}a^{-3}$, where ρ_{m0} is the present value of the matter energy density. On the other hand, the bulk viscous pressure $-\zeta\nabla_\nu u^\nu$ can be written as $-3\zeta H$. So, assuming the general equation of state for the viscous component as $P_v = w\rho_v$, the second conservation equation in (2) for the viscous fluid becomes

$$\dot{\rho}_v + 3H(1+w)\rho_v - 9\zeta H^2 = 0. \quad (3)$$

The Einstein's equations for the model are

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_v), \quad (4)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}[\rho_m + (1+3w)\rho_v - 9\zeta H].$$

We assume the parametrization for the bulk viscosity ζ of the viscous component ρ_v in the form of the following expansion:

$$\zeta = \zeta_0 + \zeta_1 H + \zeta_2 \left(\frac{\ddot{a}}{a}\right), \quad (5)$$

where ζ_0 , ζ_1 and ζ_2 are constants to be determined by the observations. The term (\ddot{a}/a) in (5) can be written as $(\ddot{a}/a) = (\ddot{a}/a)/H$. So, using second equation in (4), it can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3H}[\rho_m + (1+3w)\rho_v - 9\zeta H]. \quad (6)$$

Next we substitute eq. (6) into (5), and after re-arranging terms we obtain

$$\zeta = \frac{\zeta_0 + \zeta_1 H - \zeta_2 \left(\frac{4\pi G}{3H}\right) [\rho_m + (1+3w)\rho_v]}{1 - 9\left(\frac{4\pi G}{3}\right) \zeta_2}. \quad (7)$$

Defining the dimensionless bulk viscous coefficients as

$$\tilde{\zeta} \equiv \left(\frac{8\pi G}{3H_0}\right) \zeta, \quad \tilde{\zeta}_0 \equiv \left(\frac{24\pi G}{H_0}\right) \zeta_0$$

$$\tilde{\zeta}_1 \equiv (24\pi G) \zeta_1, \quad \tilde{\zeta}_2 \equiv \left(\frac{4\pi G}{3}\right) \zeta_2,$$

and the dimensionless Friedmann equation – first equation in (4) – as

$$E^2 \equiv \left(\frac{H}{H_0}\right)^2 = \Omega_{m0}a^{-3} + \tilde{\Omega}_v, \quad (8)$$

where $\Omega_{m0} \equiv \rho_{m0}/\rho_{\text{crit}}^0$ and $\tilde{\Omega}_v \equiv \rho_v/\rho_{\text{crit}}^0$, the equation (7) can be rewritten in dimensionless form:

$$\tilde{\zeta} = \frac{\tilde{\zeta}_0 + \tilde{\zeta}_1 E - (9\tilde{\zeta}_2/E) [\Omega_{m0}a^{-3} + (1+3w)\tilde{\Omega}_v]}{9(1-9\tilde{\zeta}_2)}, \quad (9)$$

with $\tilde{\zeta}_2 \neq 1/9$. Hence, the conservation equation for the viscous component (3) in terms of dimensionless quantities is an ordinary differential equation

$$a \frac{d\tilde{\Omega}_v}{da} + 3(1+w)\tilde{\Omega}_v - 9\tilde{\zeta}E = 0, \quad (10)$$

where E and $\tilde{\zeta}$ are given by the equations (8) and (9) respectively. Using the relation $1+z = a^{-1}$ between the redshift and the scale factor, we express the ODE (10) as

$$(1+z) \frac{d\tilde{\Omega}_v(z)}{dz} - 3(1+w)\tilde{\Omega}_v(z) + 9\tilde{\zeta} [\tilde{\Omega}_v(z) + \Omega_{m0}(1+z)^3]^{1/2} = 0. \quad (11)$$

We solve numerically this ODE assuming a dust behavior for the viscous matter (i.e., $w = 0$) and the initial conditions $\Omega_{v0} \equiv \tilde{\Omega}_v(z=0) = 0.96$, $\Omega_{m0} = 0.04$.

The dimensionless Hubble parameter $E \equiv H/H_0$ becomes

$$E(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) = \sqrt{\tilde{\Omega}_v(z) + \Omega_{m0}(1+z)^3}, \quad (12)$$

where $\tilde{\Omega}_v(z)$ is given by the numerical solution of the ODE (11), with $\tilde{\zeta}_2 \neq 1/9$. The dimensionless density parameters of the viscous dark matter $\Omega_v(z)$ and baryon $\Omega_m(z)$ components, can be expressed as

$$\Omega_v(z) = \frac{\tilde{\Omega}_v(z)}{\tilde{\Omega}_v(z) + \Omega_{m0}(1+z)^3}, \quad (13a)$$

$$\Omega_m(z) = \frac{\Omega_{m0}(1+z)^3}{\tilde{\Omega}_v(z) + \Omega_{m0}(1+z)^3}, \quad (13b)$$

where $\tilde{\Omega}_v(z)$ is given by the ODE (11). The evolution of $\Omega_v(z)$ and $\Omega_m(z)$ is shown in figure 5. For the deceleration parameter $q(a) = -(\ddot{a}/a)H^{-2}$, one obtains

$$q(z) = \frac{\Omega_{m0}(1+z)^3 + \tilde{\Omega}_v(z) - 9\tilde{\zeta}E(z)}{2(\Omega_{m0}(1+z)^3 + \tilde{\Omega}_v(z))}, \quad (14)$$

where $\tilde{\zeta}$ and $E(z)$ are given by Eqs.(9) and (12) respectively. The evolution of $q(z)$ is shown in figure 6.

III. COSMOLOGICAL PROBES

We test the viability of the model and constrain its free parameters $(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2)$ using the type Ia Supernovae

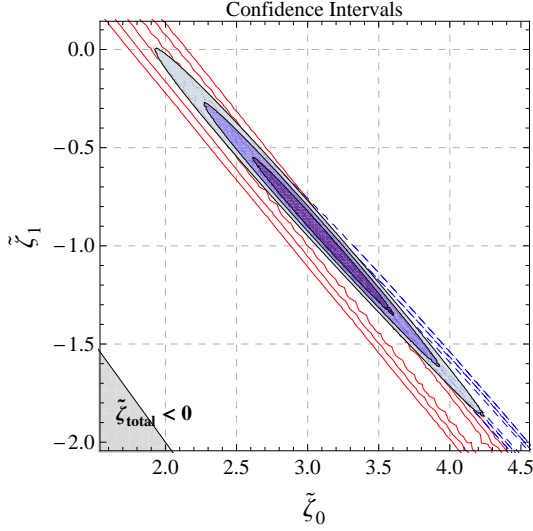


FIG. 1: Confidence intervals for the dimensionless viscous coefficients $(\tilde{\zeta}_0, \tilde{\zeta}_1)$ of a bulk viscosity parametrization of the form $\zeta = \zeta_0 + \zeta_1 H$ (we have set $\tilde{\zeta}_2 = 0$). We find as best estimates $(\tilde{\zeta}_0 = 3.12^{+0.35}_{-0.34}, \tilde{\zeta}_1 = -0.96 \pm 0.3)$, see table I. The solid red, dashed and filled contour plots correspond to the use of SNe, $H(z)$ and the joint SNe + $H(z)$ data sets respectively. The gray shaded area at the bottom left indicates the *forbidden* region where the total bulk viscosity $\tilde{\zeta}_{\text{total}}$ given by eq.(9) is *negative* at the present-day ($z = 0$). The confidence intervals shown correspond to 68.3%, 95.4% and 99.73% of probability. The Hubble constant H_0 was marginalized assuming a *constant prior* probability distribution for H_0 . It was assumed the values $\Omega_v(z = 0) = 0.96$, $\Omega_{m0} = 0.04$ and $w = 0$.

(SNe Ia) observations and the Hubble parameter $H(z)$ measured at different redshifts. We compute the *best estimated values* for pairs of $(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2)$, the *goodness-of-fit* of the model to the data and the confidence intervals by a χ^2 function minimization, to constrain their possible values with levels of statistical confidence which are shown in figures 1–3.

A. Type Ia Supernovae

We use the “Union2.1” SNe Ia data set (2012) from “The Supernova Cosmology Project” (SCP) composed by 580 type Ia supernovae [12]. The luminosity distance d_L in a flat FRW cosmology, is computed through

$$d_L(z, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, H_0) = c(1+z)H_0 \int_0^z \frac{dz'}{E(z', \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2)},$$

where $E(z, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2)$ is given by the expression (8) and c is the speed of light given in units of km/sec. The theoretical distance moduli for the k -th supernova with redshift z_k is defined as

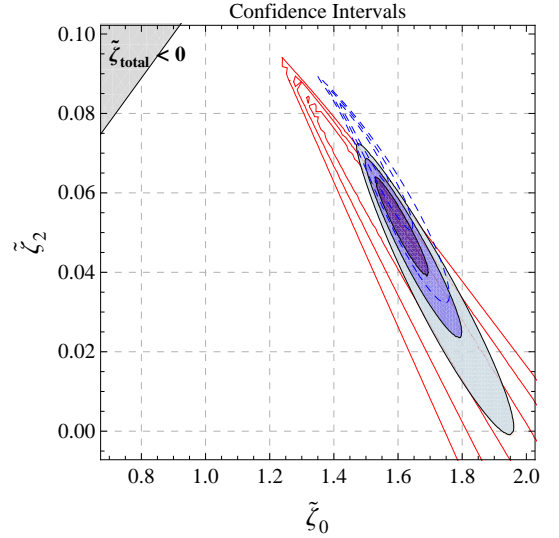


FIG. 2: Confidence intervals for $(\tilde{\zeta}_0, \tilde{\zeta}_2)$ of a bulk viscosity parametrization of the form $\zeta = \zeta_0 + \zeta_2(\ddot{a}/\dot{a})$ (we set $\tilde{\zeta}_1 = 0$). We find as best estimates $(\tilde{\zeta}_0 = 1.59^{+0.06}_{-0.05}, \tilde{\zeta}_2 = 0.05^{+0.007}_{-0.01})$, see table I. The solid red, dashed and filled contour plots correspond to the use of SNe, $H(z)$ and the joint SNe + $H(z)$ data sets respectively. The gray shaded area at the top left indicates the *forbidden* region where the total bulk viscosity $\tilde{\zeta}_{\text{total}}$ given by eq.(9) is *negative* at the present-day ($z = 0$). The confidence intervals shown correspond to 68.3%, 95.4% and 99.73% of probability.

$$\mu^t(z_k, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, H_0) \equiv m - M = 5 \log_{10} \left[\frac{d_L(z_k, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2, H_0)}{\text{Mpc}} \right] + 25, \quad (15)$$

where m and M are the apparent and absolute magnitudes of the SNe Ia respectively, and the superscript ‘t’ stands for “*theoretical*”. We construct the statistical χ^2 function as

$$\chi_{\text{SNe}}^2(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) \equiv \sum_{k=1}^n \frac{[\mu^t(z_k, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) - \mu_k]^2}{\sigma_k^2}, \quad (16)$$

where μ_k is the *observational* distance moduli for the k -th supernova, σ_k^2 is the variance of the measurement and n is the amount of supernova in the data set ($n = 580$). The results are shown in table I and the confidence intervals for the pairs $\tilde{\zeta}_0$ vs $\tilde{\zeta}_1$, $\tilde{\zeta}_0$ vs $\tilde{\zeta}_2$ and $\tilde{\zeta}_1$ vs $\tilde{\zeta}_2$ are shown in figures 1 to 3 respectively. The Hubble constant H_0 is marginalized assuming a *constant prior* distribution (see appendix A of [8]).

B. Hubble expansion rate

For the Hubble parameter $H(z)$ measured at different redshifts, we use the 12 data listed in table 2 of Busca

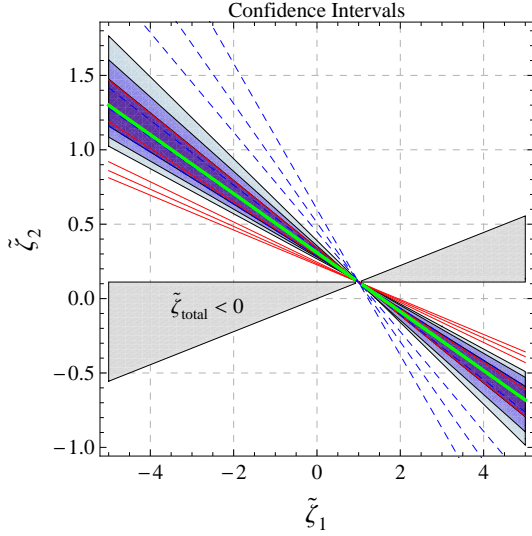


FIG. 3: Confidence intervals for $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ of a bulk viscosity parametrization of the form $\zeta = \zeta_1 H + \zeta_2 (\ddot{a}/\dot{a})$ (we set $\tilde{\zeta}_0 = 0$). The solid red, dashed and filled contour plots correspond to the use of SNe, $H(z)$ and the joint SNe + $H(z)$ data sets respectively. The confidence intervals shown correspond to 68.3%, 95.4% and 99.73% of probability. The best estimated values for $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ are *all* those points that lie on the line equation $\tilde{\zeta}_2 = m\tilde{\zeta}_1 + b$, where $m = -0.19808$, $b = 0.30918$ with $\chi^2_{\min} = 639.131$ (green line) when using the combined SNe + $H(z)$ data sets, except at the singular point $(\tilde{\zeta}_1 = 1, \tilde{\zeta}_2 = 1/9)$ that corresponds to the vertex visible in the figure. The gray shaded area indicates the *forbidden* region where the total bulk viscosity $\tilde{\zeta}_{\text{total}}$ given by eq.(9) is *negative* at the present day ($z = 0$).

et al. (2012) [14], where 11 data come from references [15]–[17]. The value $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, is assumed for the data of Blake et al. (2011) [15] as Busca et al. suggest. The χ^2 function is defined as

$$\chi^2_{\text{H}}(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) = \sum_i^{12} \left(\frac{H(z_i, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) - H_i^{\text{obs}}}{\sigma_{H_i}} \right)^2, \quad (17)$$

where $H(z_i, \tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2)$ and H_i^{obs} are the theoretical and observed values respectively and σ_{H_i} the standard deviation of each H_i^{obs} entry.

The *total* χ^2_{t} function which combines the SNe and $H(z)$ data sets together, is chosen in the following way:

$$\chi^2_{\text{t}} = \chi^2_{\text{SNe}} + \chi^2_{\text{H}},$$

where χ^2_{SNe} and χ^2_{H} are given by expressions (16) and (17) respectively. The function χ^2_{t} is then numerically minimised in order to compute the “*best estimates*” for pairs of the viscous coefficients: $(\tilde{\zeta}_0, \tilde{\zeta}_1)$, $(\tilde{\zeta}_0, \tilde{\zeta}_2)$, and $(\tilde{\zeta}_1, \tilde{\zeta}_2)$, where the remaining viscous coefficient in each case is assumed to vanish. The minimum value of the χ^2 function gives the best estimated values of the pairs $(\tilde{\zeta}_0, \tilde{\zeta}_1)$, $(\tilde{\zeta}_0, \tilde{\zeta}_2)$, and $(\tilde{\zeta}_1, \tilde{\zeta}_2)$, and measures the goodness-of-fit of the model to data.

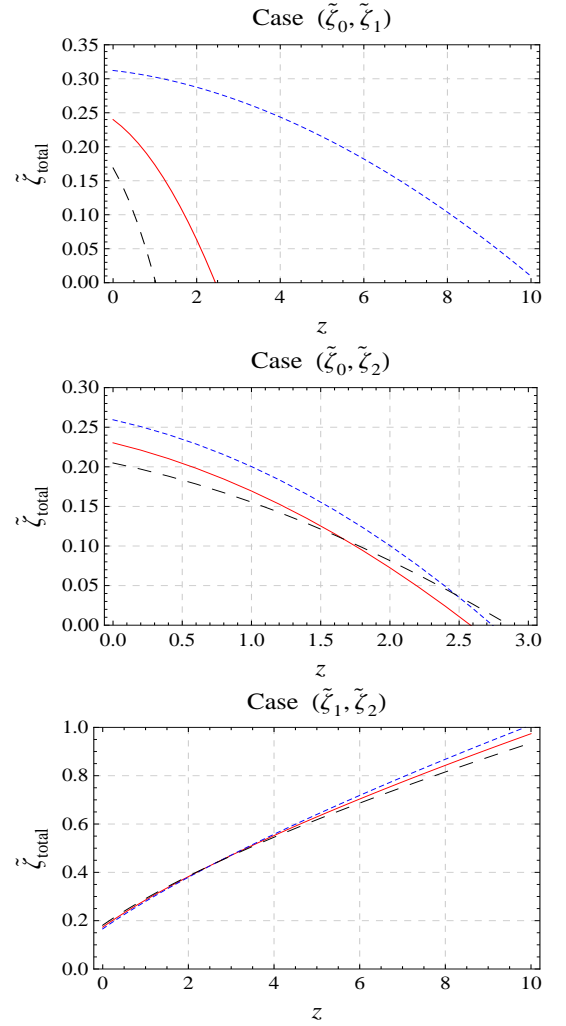


FIG. 4: Evolution of the *total* bulk viscosity $\tilde{\zeta}_{\text{total}}$ with respect to the redshift z , for the different cases. The plots come from the evaluation of the equation (9) at the best estimated values for $(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2)$ and error values. In the upper and middle panels the red solid lines correspond to the evaluation of eq. (9) at $(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) = (3.12, -0.96, 0)$ and $(1.59, 0, 0.05)$, respectively. The short dashed blue lines correspond to evaluation at $(3.47, -0.66, 0)$ and $(1.65, 0, 0.057)$ respectively, while the long dashed black lines correspond to $(2.78, -1.26, 0)$ and $(1.54, 0, 0.04)$ respectively. In the bottom panel the best estimated values for $(\tilde{\zeta}_1, \tilde{\zeta}_2)$, with $\tilde{\zeta}_0 = 0$, correspond to *all* those points which lie on the line $\tilde{\zeta}_2 = m\tilde{\zeta}_1 + b$, where $m = -0.19808$, $b = 0.30918$ (see figure 3). It turns out that the evolution of $\tilde{\zeta}_{\text{total}}(z)$ is *independent* of the values of $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ when these are located at the best estimated *line* indicated above (the green line at figure 3). The solid red line in the bottom panel corresponds to the evaluation of eq. (9) at some arbitrary values of $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ related by the line equation given above. The short dashed blue, and long dashed black lines correspond to the evaluation at the limits of the 1σ confidence contour shown on figure 3.

Viscous model

Best estimates	Assumption	χ^2_{\min}	$\chi^2_{\text{d.o.f.}}$
$\tilde{\zeta}_0 = 3.12^{+0.35}_{-0.34}$ $\tilde{\zeta}_1 = -0.96 \pm 0.3$	$\tilde{\zeta}_2 = 0$	573.34	0.97
$\tilde{\zeta}_0 = 1.59^{+0.06}_{-0.05}$ $\tilde{\zeta}_2 = 0.05^{+0.007}_{-0.01}$	$\tilde{\zeta}_1 = 0$	573.34	0.97

TABLE I: Best estimated values of the dimensionless viscous coefficients ($\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2$) for the dark component with bulk viscosity parametrized as $\zeta = \zeta_0 + \zeta_1 H + \zeta_2 (\ddot{a}/\dot{a})$. It was assumed $w = 0$ and $\tilde{\Omega}_v(z = 0) = 0.96$ for the dark component, and $\Omega_{m0} = 0.04$ for the baryon matter. The best estimates were computed using the SNe + $H(z)$ data sets. The first two columns show the best estimated values for pairs of viscous coefficients ($\tilde{\zeta}_i, \tilde{\zeta}_j$), where the remaining viscous coefficient $\tilde{\zeta}_k$ in each case, is set to zero (third column). The fourth and fifth columns show the minimum of the χ^2 function and its corresponding “ χ^2 function by *degrees of freedom*”: $\chi^2_{\text{d.o.f.}} \equiv \chi^2_{\min}/(n - p)$, where n is the number of data, and p is the number of free parameters estimated. The errors in the estimations are given to 1σ . The best estimated values for $(\tilde{\zeta}_1, \tilde{\zeta}_2)$, with $\tilde{\zeta}_0 = 0$, correspond to *all* those points which lie on the line $\tilde{\zeta}_2 = m\tilde{\zeta}_1 + b$ ($m = -0.19808$, $b = 0.30918$), with $\chi^2_{\min} = 639.131$ ($\chi^2_{\text{d.o.f.}} = 1.08$). Figures 1 to 3 show the confidence intervals. The Hubble constant H_0 was marginalized assuming a *constant* prior distribution.

The definition of “ χ^2 function by *degrees of freedom*”: $\chi^2_{\text{d.o.f.}} \equiv \chi^2_{\min}/(n - p)$, where n is the number of *total* combined data used, and p is the number of free parameters estimated, is also used in our computations.

C. Local Second Law of Thermodynamics

The *local* entropy production for a fluid on a FRW spacetime is expressed as [9]

$$T \nabla_\nu s^\nu = \zeta (\nabla_\nu u^\nu)^2 = 9H^2 \zeta, \quad (18)$$

where T is the temperature of the fluid, $\nabla_\nu s^\nu$ is the rate of entropy production in a unit volume, and ζ is the *total* bulk viscosity. The second law of the thermodynamics can be stated as $T \nabla_\nu s^\nu \geq 0$. Hence, since the Hubble parameter H is positive for an expanding Universe, Eq. (18) implies that $\tilde{\zeta} \geq 0$, where $\tilde{\zeta}$ is given by the expression (9). This inequality is an additional constraint in the possible values for the *total* dimensionless viscous parameter in our model.

IV. DYNAMICAL SYSTEMS

The dynamical systems tools offer a powerful mean to extract relevant information out of the given cosmological model by investigating the equivalent phase space. Critical points in the phase space: past/future attractors, saddle points, etc., can be correlated with generic solutions of the cosmological field equations. In order to

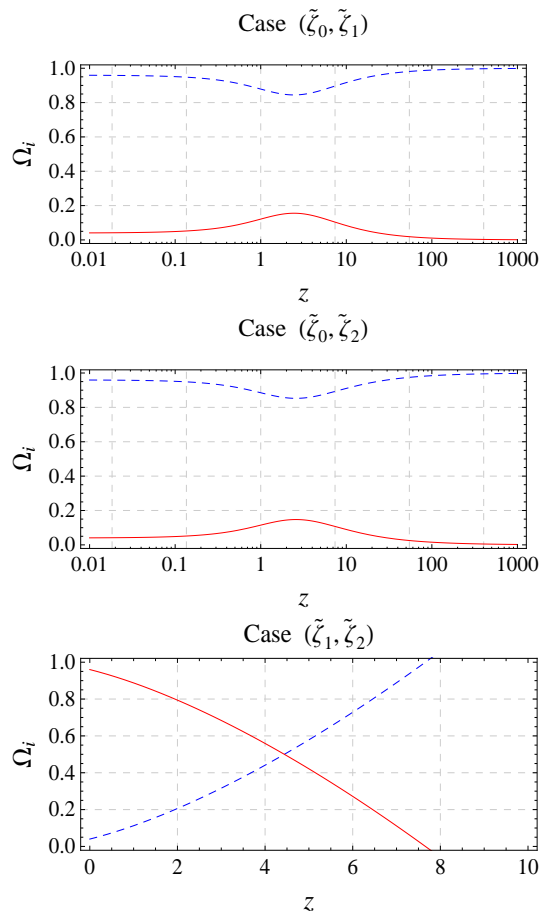


FIG. 5: Evolution of the dimensionless density parameters of the viscous dark matter component $\Omega_v(z)$ [solid red line] and of the baryon matter component $\Omega_m(z)$ [dashed blue line] with respect to the redshift z , for the different cases. In the top and middle panels the curves correspond to the evaluation of eqs. (13) at the best estimated values $(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) = (3.12, -0.96, 0)$ and $(1.59, 0, 0.05)$, respectively. In the bottom panel the best estimated values for $(\tilde{\zeta}_1, \tilde{\zeta}_2)$, correspond to *all* those points which lie on the line: $\tilde{\zeta}_2 = m\tilde{\zeta}_1 + b$, where $m = -0.19808$, $b = 0.30918$ (see figure 3). It turns out that the evolution of $\Omega_v(z)$ is also *independent* of the values of $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ when these values are located at the best estimated *line* indicated above. The curves in the bottom panel correspond to the evaluation of eqs. (13) at some arbitrary values of $(\tilde{\zeta}_1, \tilde{\zeta}_2)$ related by the line equation given above. It is seen that starting at some redshift $z \sim 7.7$, $\Omega_v(z) > 1$, while $\Omega_v(z) < 0$, which is a nonsense. From the top and middle panels it is seen that $\Omega_v(z)$ is always the dominant component with respect to $\Omega_m(z)$ at late times.

be able to apply these tools one has to follow the steps enumerated here: i) to identify the phase space variables that allow writing the system of cosmological equations

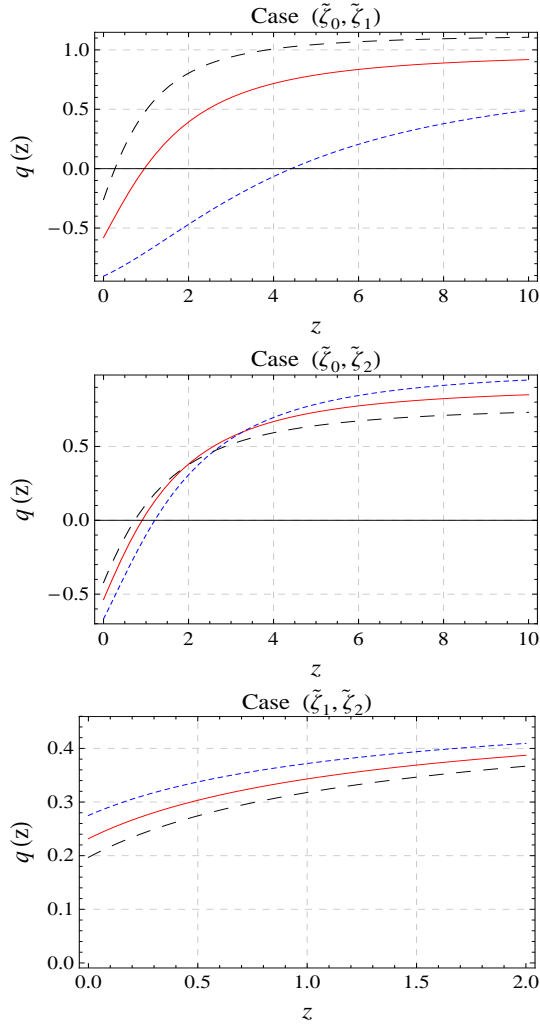


FIG. 6: Evolution of the deceleration parameter $q(z)$ with respect to the redshift z for the different cases evaluated at the best estimated values. In the top and middle panels the solid red lines correspond to the evaluation of eq. (14) at $(\tilde{\zeta}_0, \tilde{\zeta}_1, \tilde{\zeta}_2) = (3.12, -0.96, 0)$ and $(1.59, 0, 0.05)$, respectively, while the short dashed blue lines correspond to evaluation at $(3.47, -0.66, 0)$ and $(1.65, 0, 0.057)$ respectively. The long dashed black lines correspond to $(2.78, -1.26, 0)$ and $(1.54, 0, 0.04)$ respectively. In the bottom panel the best estimated values for $(\tilde{\zeta}_1, \tilde{\zeta}_2)$, with $\tilde{\zeta}_0 = 0$, correspond to *all* those points which lie on the line: $\tilde{\zeta}_2 = m\tilde{\zeta}_1 + b$, where $m = -0.19808$, $b = 0.30918$. The evolution of $q(z)$ is independent of the best estimated values of $(\tilde{\zeta}_1, \tilde{\zeta}_2)$. The solid red line in the bottom panel corresponds to the evaluation of eq. (14) at some arbitrary values of $(\tilde{\zeta}_1, m\tilde{\zeta}_1 + b)$, while the short dashed blue and long dashed black lines correspond to the evaluation at the limits of the 1σ confidence contour shown on figure 3.

in the form of an autonomous system of ODE,¹ ii) with the help of the chosen phase space variables to build an autonomous system of ODE out of the original system of cosmological equations, and an usually forgotten or unappreciated step, iii) to identify the phase space spanned by the chosen variables that is relevant to the cosmological model under study.

Our goal in this section is to write the cosmological equations of the model (2-4):²

$$\begin{aligned}\dot{\rho}_m + 3H\rho_m &= 0, \\ \dot{\rho}_v + 3H(1+\omega)\rho_v - 9\zeta H^2 &= 0, \\ 3H^2 &= \rho_v + \rho_m, \\ 6\dot{H} + 6H^2 &= -\rho_m - (1+3\omega)\rho_v + 9\zeta H, \quad (19)\end{aligned}$$

in the form of an autonomous system of ODE. To this end we have to choose appropriate phase space variables. In the present case our starting phase space variable is the dimensionless energy density parameter of the viscous fluid:

$$x \equiv \Omega_v = \frac{\rho_v}{3H^2}, \quad 0 \leq x \leq 1. \quad (20)$$

In terms of this variable the Friedmann constraint (third equation in (19)) can be written as $\Omega_m = 1 - x$, where we use the standard definition of the dimensionless energy density parameter of the i -th matter component, $\Omega_i \equiv \rho_i/3H^2$. Also, one can write the following autonomous ODE:

$$x' = -3(1+\omega)x - 2x\frac{H'}{H} + 3\frac{\zeta}{H}, \quad (21)$$

or, since

$$2\frac{H'}{H} = -3(1+\omega x) + 3\frac{\zeta}{H}, \quad (22)$$

the former equation can be written in more compact form:

$$x' = 3(x-1)\left(\omega x - \frac{\zeta}{H}\right). \quad (23)$$

In the above equations the tilde accounts for derivative with respect to the parameter, $\tau = \ln a$.

As already mentioned, here we shall investigate a viscous coefficient of the form given in equation (5):

$$\zeta = \zeta_0 + \zeta_1 H + \zeta_2 \frac{\ddot{a}}{\dot{a}},$$

¹ There might be several different possible choices, however, not all of them allow for the minimum possible dimensionality of the phase space.

² Here we use units where $8\pi G = 1$.

or, since $\ddot{a}/a = \dot{H} + H^2$, then the viscous parameter will obey the following equation:

$$\begin{aligned}\frac{\zeta}{H} &= \frac{\zeta_0}{H} + \zeta_1 + \zeta_2 + \frac{H'}{H}\zeta_2 \\ &= \frac{2\zeta_0/H + 2\zeta_1 - (1 + 3\omega x)\zeta_2}{2 - 3\zeta_2},\end{aligned}\quad (24)$$

where, in the last row, we have taken into consideration equation (22). If we substitute back Eq.(24) into (23), we obtain the following master ODE:

$$x' = \frac{6(x-1)}{2-3\zeta_2} \left(\omega x - \frac{\zeta_0}{H} - \zeta_1 + \frac{\zeta_2}{2} \right). \quad (25)$$

Several cosmological parameters, such as the deceleration parameter $q = -1 - H'/H$, and the equation of state (EoS) effective parameter $\omega_{eff} = -1 - 2H'/3H$, can also be rewritten in terms of the variable x . In fact, if take into account equations (22) and (24) one obtains:

$$\begin{aligned}q &= \frac{1 + 3\omega x - 3\zeta_0/H - 3\zeta_1}{2 - 3\zeta_2}, \\ \omega_{eff} &= \frac{2\omega x - 2\zeta_0/H - 2\zeta_1 + \zeta_2}{2 - 3\zeta_2},\end{aligned}\quad (26)$$

respectively.

Depending on the case under consideration one would need yet another phase space variable which would be related with the viscous coefficient ζ_0 (see below). In what follows we shall split the dynamical systems study into two different cases: A) when the viscous EoS parameter $\omega \neq 0$, and B) when the viscous fluid is dust: $\omega = 0$.

A. Viscous EoS $\omega \neq 0$

1. Case with $\zeta_0 = 0$.

If we set $\zeta_0 = 0$ in Eq.(25), we obtain the following autonomous ODE for this particular case:

$$x' = \frac{6\omega(x-1)}{2-3\zeta_2} \left(x - \frac{2\zeta_1 - \zeta_2}{2\omega} \right). \quad (27)$$

The phase space is the segment, $\Psi = \{x|0 \leq x \leq 1\}$.

Two equilibrium/critical points are found.

1. The first one,

$$P_v : \left(x = \frac{\rho_v}{3H^2} = 1 \right),$$

corresponds to the viscous matter-dominated solution. The deceleration and EoS effective parameters, in this case, are given by:

$$q = \frac{1 + 3(\omega - \zeta_1)}{2 - 3\zeta_2}, \quad \omega_{eff} = \frac{2(\omega - \zeta_1) + \zeta_2}{2 - 3\zeta_2}.$$

The solution corresponds to accelerated expansion whenever, either $\zeta_1 > (1+3\omega)/3$, $\zeta_2 < 2/3$, or, $\zeta_1 < (1+3\omega)/3$, $\zeta_2 > 2/3$. Otherwise it will correspond to decelerated expansion instead.

If consider small τ -dependent perturbation $\epsilon = \epsilon(\tau)$ around this critical point: $x \rightarrow 1 + \epsilon(\tau)$, up to $\mathcal{O}(\epsilon^2)$, the perturbation will obey the following linearized ODE:

$$\epsilon'(\tau) = 3 \frac{2\omega - 2\zeta_1 + \zeta_2}{2 - 3\zeta_2} \epsilon(\tau),$$

which can be readily integrated,

$$\epsilon(\tau) = \epsilon_0 e^{\lambda\tau}, \quad \lambda = 3 \frac{\zeta_2 - 2\zeta_1 + 2\omega}{2 - 3\zeta_2},$$

where ϵ_0 is an integration constant. The solution is stable or, in other words, it is a future attractor in the phase segment if $\lambda < 0$, i. e., if either

$$2\zeta_1 - \zeta_2 > 2\omega, \quad \zeta_2 < 2/3,$$

or

$$2\zeta_1 - \zeta_2 < 2\omega, \quad \zeta_2 > 2/3,$$

which coincide with the regions in the space of parameters $(\zeta_1, \zeta_2, \omega)$ where the point P_v is correlated with inflationary expansion ($q < 0$). Otherwise, if

$$2\zeta_1 - \zeta_2 < 2\omega, \quad \zeta_2 < 2/3,$$

or

$$2\zeta_1 - \zeta_2 > 2\omega, \quad \zeta_2 > 2/3,$$

the viscous matter-dominated solution is unstable (it is a past attractor), while the expansion occurs at a decelerated pace. Hence, either the viscous matter-dominated critical point P_v ($3H^2 = \rho_v$), is the future inflationary attractor/end-point of any phase space orbit, or, alternatively, it is the past attractor/source point in the phase space, which is associated with decelerated expansion.

2. The second critical point,

$$P_{m/v} : \left(x = \frac{2\zeta_1 - \zeta_2}{2\omega} \right) \Rightarrow 3H^2 = \frac{2\omega\rho_v}{2\zeta_1 - \zeta_2},$$

exists whenever³ $0 < 2\zeta_1 - \zeta_2 \leq 2\omega$, and corresponds to matter/viscous matter-scaling solution:

$$\frac{\Omega_m}{\Omega_v} = \frac{2\omega - 2\zeta_1 + \zeta_2}{2\zeta_1 - \zeta_2}.$$

³ We shall be assuming that ω is a non-negative quantity, which covers the most interesting physical situations.

The deceleration and the EoS parameters for this critical point are: $q = 1/2$ and $\omega_{eff} = 0$ respectively.

Small τ -dependent perturbations around this critical point will obey, $\epsilon' = \lambda\epsilon$, or after integration,

$$\epsilon(\tau) = \epsilon_0 e^{\lambda\tau}, \quad \lambda = \frac{2\zeta_1 - \zeta_2 - 2\omega}{2 - 3\zeta_2}.$$

Hence the scaling critical point is stable, i. e., it is the future attractor in the phase space, if $\zeta_2 < 2/3$. Otherwise, if $\zeta_2 > 2/3$, this equilibrium point is the past attractor. In consequence, when the scaling equilibrium point exists, either it is the future attractor while the first equilibrium point ($x = 1$) is the past attractor, or vice versa.

In this case ($\zeta_0 = 0$), if $2\zeta_1 - \zeta_2 < 2\omega$, then the orbits in the phase segment either depart from the viscous matter-dominated, decelerated solution, and end-up at the, also decelerated, scaling solution, or they are repelled from the scaling equilibrium point (corresponding to decelerated expansion always) and are attracted towards the viscous matter-dominated inflationary solution. In either case none of these scenarios is suitable to accommodate the present cosmological paradigm, since there is no any critical point that could be associated with conventional matter and/or radiation dominance, which are included here in the matter component characterized by energy density ρ_m .⁴ Radiation and matter-dominated phases are necessary to explain the formation of the amount of cosmic structure we see, in particular the right growth of structure [13].

An interesting situation occurs when $\zeta_2 = 0$, $\zeta_1 \neq 0$. In this case, for the viscous matter-dominated solution $x = 1$, one has

$$q = \frac{1 + 3(\omega - \zeta_1)}{2}, \quad \omega_{eff} = \omega - \zeta_1,$$

so that the solution is inflationary if $\zeta_1 > (1 + 3\omega)/3$. Besides, this critical point is a future attractor in the phase space (segment) only if $\zeta_1 > \omega$. Hence, this solution is stable and inflationary only if $\zeta_1 > \omega + 1/3$.

Alternatively, the scaling equilibrium point $x = \zeta_1/\omega$,

$$\frac{\Omega_m}{\Omega_v} = \frac{\omega - \zeta_1}{\zeta_1}, \quad q = 1/2, \quad \omega_{eff} = 0,$$

exists whenever, $0 \leq \zeta_1 \leq \omega$. It is stable whenever it exists, $\zeta_1 < \omega$. Hence, when both critical points coexist, the viscous matter-dominated (decelerated) solution is

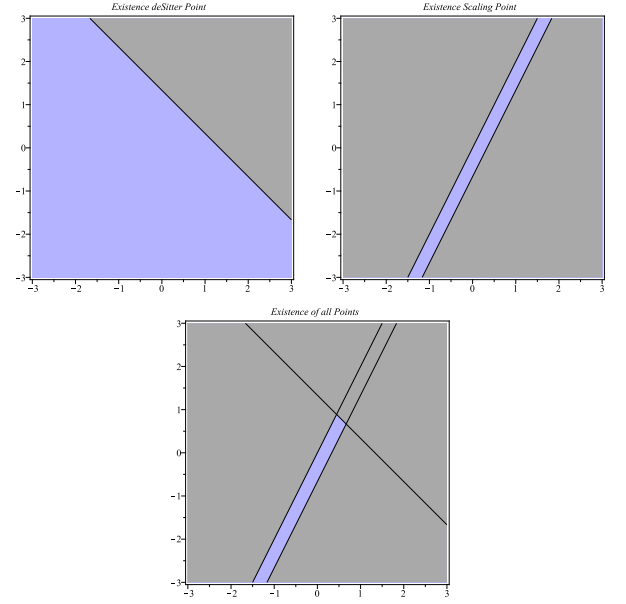


FIG. 7: Existence of the critical points in parameters space (ζ_1, ζ_2) for the case IV A 2. Regions in which the given(s) critical point(s) exists are the blue-coloured ones. The parameter ω has been arbitrarily chosen to be $\omega = 1/3$ (viscous "radiation").

the past attractor, while the (also decelerating) scaling solution is the future attractor. Curiously, if $\zeta_1 > \omega + 1$, i. e., if the viscous matter-dominated solution is the future attractor (besides, it is the only critical point in the phase segment), the effective EoS parameter behaves like a phantom, $\omega_{eff} < -1$.

If, on the contrary, $\zeta_1 = 0$, $\zeta_2 \neq 0$, then for the viscous matter-dominated critical point $x = 1$,

$$q = \frac{1 + 3\omega}{2 - 3\zeta_2}, \quad \omega_{eff} = \frac{2\omega + \zeta_2}{2 - 3\zeta_2},$$

so that this solution corresponds to inflationary expansion if $\zeta_2 > 2/3$. It is stable when, either, $\zeta_2 + 2\omega < 0$, $\zeta_2 < 2/3$, or, $\zeta_2 + 2\omega > 0$, $\zeta_2 > 2/3$. The scaling solution, $x = -\zeta_2/2\omega$,

$$\frac{\Omega_m}{\Omega_v} = -\frac{2\omega + \zeta_2}{\zeta_2},$$

exists if $-2\omega \leq \zeta_2 < 0$. It is stable whenever it exists. In this case the viscous matter-dominated cosmic fluid mimics phantom behavior if $\zeta_2 > \omega + 1$ ($\omega_{eff} < -1$).

As before, no critical point in the phase space can be associated with conventional matter dominance as required by the standard cosmological paradigm.

⁴ As a matter of fact, in the present work, for simplicity, it has been assumed that the conventional matter behaves like pressureless dust, but it is clear that even if consider it to be radiation there would not be any critical point associated with radiation-domination (see the next section).

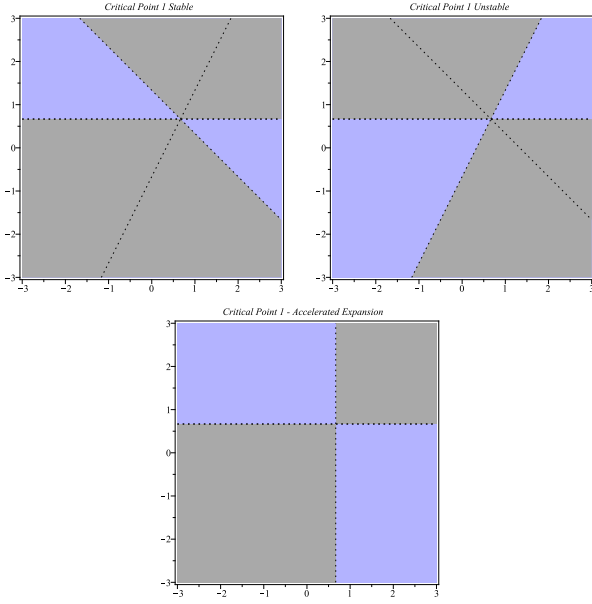


FIG. 8: In the upper panels the stability of the viscous matter-dominated critical point P_v for case IV A 2 is shown in the space of parameters (ζ_1, ζ_2) (as before ζ_2 stands for the vertical axis while ζ_1 is the horizontal axis, and $\omega = 1/3$). The critical point P_v is stable in the blue-coloured regions in the upper left-hand figure, while, it is unstable in the blue-coloured regions in the upper right-hand figure. In the lower panel the regions in parameter space where the critical point P_v corresponds to inflationary phase are shown (also coloured in blue).

2. Case with $\zeta_0 \neq 0$.

In this case to the already existing variable x one has to add a new one:

$$y = \frac{1}{\zeta_0/H + 1} \Rightarrow \frac{\zeta_0}{H} = \frac{1-y}{y}.$$

Hence, the phase space is the bounded plane region

$$\Psi = \{(x, y) | 0 \leq x \leq 1, 0 < y \leq 1\}.$$

As before, the Friedmann constraint reads $\Omega_m = 1 - x$. The corresponding autonomous system of ODE looks like

$$\begin{aligned} x' &= \frac{3(2 - 2\zeta_1 + \zeta_2)}{(2 - 3\zeta_2)y} (x - 1) \left(y + \frac{2\omega xy - 2}{2 - 2\zeta_1 + \zeta_2} \right), \\ y' &= \frac{3(2 - \zeta_1 - \zeta_2)}{2 - 3\zeta_2} (y - 1) \left(y + \frac{\omega xy - 1}{2 - \zeta_1 - \zeta_2} \right). \end{aligned} \quad (28)$$

For the deceleration and EoS parameters the following expressions are obtained:

$$\begin{aligned} q &= \frac{(4 - 3\zeta_1)y + 3\omega xy - 3}{(2 - 3\zeta_2)y}, \\ \omega_{eff} &= \frac{(2 - 2\zeta_1 + \zeta_2)y + 2\omega xy - 2}{(2 - 3\zeta_2)y}. \end{aligned} \quad (29)$$

The critical points of (28), $P_i : (x_i, y_i)$, together with their main properties are summarized as follows.

1. Viscous matter-dominated solution $P_v : (1, 1)$. This case corresponds either to the formal limit $\zeta_0 = 0$, or to the initial singular state characterized by $H \rightarrow \infty$. We have

$$q = \frac{1 - 3\zeta_1 + 3\omega}{2 - 3\zeta_2}, \quad \omega_{eff} = \frac{\zeta_2 - 2\zeta_1 + 2\omega}{2 - 3\zeta_2}.$$

The eigenvalues of the linearization (Jacobian) matrix for this point are:

$$\begin{aligned} \lambda_1 &= \frac{3(1 - \zeta_1 - \zeta_2 + \omega)}{2 - 3\zeta_2}, \\ \lambda_2 &= \frac{3(\zeta_2 - 2\zeta_1 + 2\omega)}{2 - 3\zeta_2}. \end{aligned}$$

This solution is a past attractor if, either $\zeta_1 < \omega + 1/3$, $\zeta_2 < 2/3$, or $\zeta_1 > \omega + 1/3$, $\zeta_2 > 2/3$. The stability properties of this point in the space of parameters (ζ_1, ζ_2) , are shown in the figure 8, where the free parameter ω has been arbitrarily set equal to $1/3$ ("viscous radiation").

2. de Sitter (also viscous fluid-dominated) solution

$$\begin{aligned} P_{dS} : \left(1, \frac{1}{2 - \zeta_1 - \zeta_2 + \omega} \right) &\Rightarrow \\ H = H_0 = \frac{\zeta_0}{1 - \zeta_1 - \zeta_2 + \omega}, \end{aligned}$$

characterized by $q = -1$, $\omega_{eff} = -1$. The eigenvalues of the Jacobian matrix corresponding to this critical point are:

$$\begin{aligned} \lambda_1 &= -\frac{3(1 - \zeta_1 - \zeta_2 + \omega)}{2 - 3\zeta_2}, \\ \lambda_2 &= -\frac{3}{(2 - \zeta_1 - \zeta_2 + \omega)^2}. \end{aligned}$$

The de Sitter solution P_{dS} exists whenever $0 < y \leq 1$, i. e., if: $\zeta_1 + \zeta_2 \leq \omega + 1$ (see Fig.7). It is the future attractor in Ψ if: $\zeta_1 + \zeta_2 < \omega + 1$, $\zeta_2 < 2/3$. Otherwise, if: $\zeta_1 + \zeta_2 < \omega + 1$, $\zeta_2 > 2/3$, P_{dS} is a saddle critical point instead. The stability of this equilibrium point in the space of parameters (ζ_1, ζ_2) is shown in Fig.9, where the additional free parameter ω has been arbitrarily chosen: $\omega = 1/3$.

3. Matter/viscous matter-scaling solution,

$$P_{m/v} : \left(\frac{2\zeta_1 - \zeta_2}{2\omega}, 1 \right), \quad \frac{\Omega_m}{\Omega_v} = \frac{2\omega - 2\zeta_1 + \zeta_2}{2\zeta_1 - \zeta_2}.$$

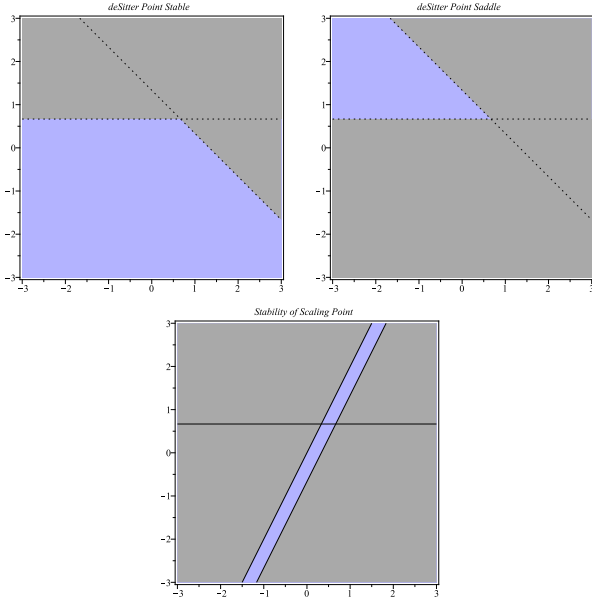


FIG. 9: Stability of the de Sitter critical point in the space of parameters (ζ_1, ζ_2) , for case IV A 2, is shown in the upper panels while, in the lower panel the stability of the scaling point is depicted. The regions of interest in each case are the ones blue-coloured.

This critical point exists ($0 \leq x \leq 1$) if (see Fig.7),

$$\frac{\zeta_2}{2} \leq \zeta_1 \leq \frac{\zeta_2}{2} + \omega.$$

It is characterized by, $q = 1/2$, $\omega_{eff} = 0$. The eigenvalues of the corresponding Jacobian matrix are

$$\lambda_1 = 3/2, \lambda_2 = -\frac{3(\zeta_2 - 2\zeta_1 + 2\omega)}{2 - 3\zeta_2},$$

so that, if, $2\zeta_1 - \zeta_2 < 2\omega$, $\zeta_2 < 2/3$, it is a saddle point in the phase space. Otherwise, if, $2\zeta_1 - \zeta_2 < 2\omega$, $\zeta_2 > 2/3$, it is the past attractor instead. In Fig.9, the stability of the scaling point is depicted in the space of parameters (ζ_1, ζ_2) . As before viscous radiation has been considered ($\omega = 1/3$).

B. Pressureless Viscous Fluid ($\omega = 0$).

In this case equations (28) simplify to:

$$\begin{aligned} x' &= \frac{3(2 - 2\zeta_1 + \zeta_2)}{2 - 3\zeta_2} \left(\frac{x-1}{y} \right) \left(y - \frac{1}{1 - \zeta_1 + \zeta_2/2} \right), \\ y' &= \frac{3(2 - \zeta_1 - \zeta_2)}{2 - 3\zeta_2} (y-1) \left(y - \frac{1}{2 - \zeta_1 - \zeta_2} \right). \end{aligned} \quad (30)$$

Besides,

$$q = \frac{(4 - 3\zeta_1)y - 3}{(2 - 3\zeta_2)y}, \quad \omega_{eff} = \frac{(1 - \zeta_1 + \zeta_2/2)y - 1}{2(2 - 3\zeta_2)y}.$$

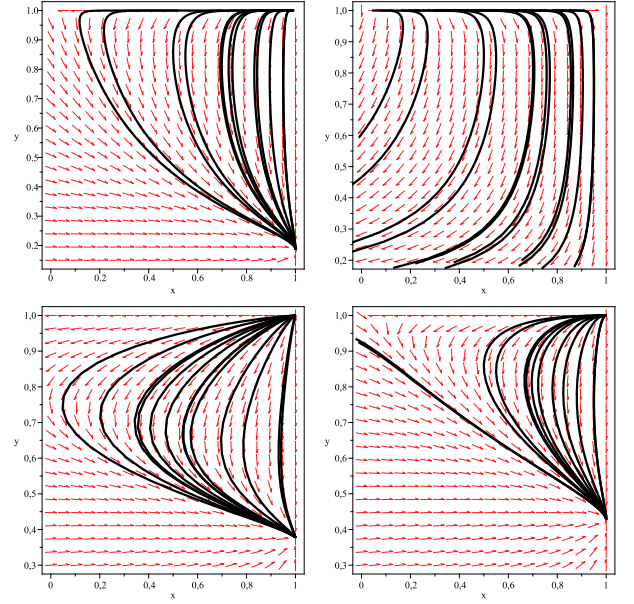


FIG. 10: Phase portraits (x, y) for the case IV A 2, for different choices of the free parameters ζ_1 , and ζ_2 ($\omega = 1/3$). The following values of the free parameters have been chosen in the upper panels: $\zeta_1 = -1$, $\zeta_2 = -2$ – left-hand panel, and $\zeta_1 = 2$, $\zeta_2 = 4$ – right-hand panel. It is seen that, while in the latter case the de Sitter point does not exist (the past attractor/source point is the scaling solution and the viscous matter-dominated phase is a saddle point), in the former case, the viscous matter point is the past attractor, the scaling solution is a saddle critical point and the de Sitter phase is the future attractor. In the lower panels we have taken best estimated values of $\zeta_1 = \tilde{\zeta}_1/3$, and $\zeta_2 = 6\tilde{\zeta}_2$ from table I: $\tilde{\zeta}_1 = -0.96$, $\tilde{\zeta}_2 = 0$ – left-hand panel, and $\tilde{\zeta}_1 = 0$, $\tilde{\zeta}_2 = 0.05$ – right-hand panel. In both cases the orbits are repelled from the viscous matter-dominated solution and approach to the de Sitter future attractor. The scaling critical point does not exist.

Only two critical points of the autonomous system of ODE (30) are found:

1. Viscous matter-dominated solution $P_v : (1, 1)$. The relevant parameters are:

$$q = \frac{1 - 3\zeta_1}{2 - 3\zeta_2}, \quad \omega_{eff} = \frac{\zeta_2 - 2\zeta_1}{2 - 3\zeta_2},$$

while the eigenvalues of the linearization matrix are:

$$\lambda_1 = \frac{3(\zeta_2 - 2\zeta_1)}{2 - 3\zeta_2}, \quad \lambda_2 = \frac{3(1 - \zeta_1 - \zeta_2)}{2 - 3\zeta_2}.$$

The stability of this point in the space of parameters (ζ_1, ζ_2) , is shown in the figure 11.

2. de Sitter equilibrium point

$$P_{dS} : \left(1, \frac{1}{2 - \zeta_1 - \zeta_2} \right), \quad q = -1, \quad \omega_{eff} = -1,$$

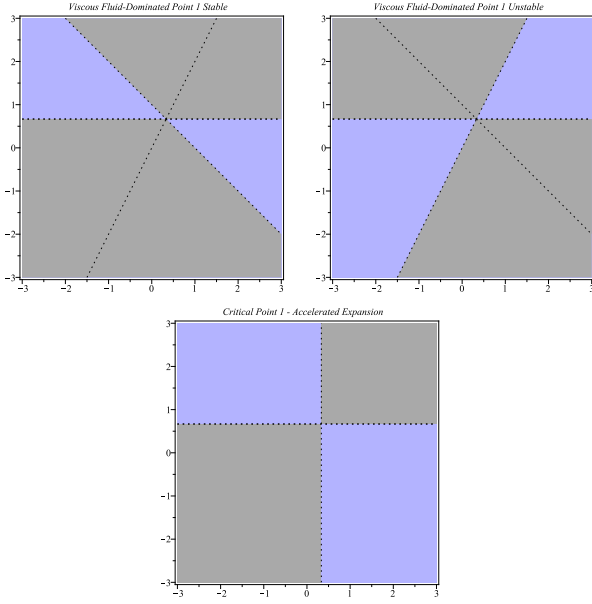


FIG. 11: Stability of the viscous matter-dominated critical point in (ζ_1, ζ_2) -space for the case IV B (upper panels). In the lower panel the regions in the parameters space where this critical point is associated with inflationary expansion are shown. As before the regions of interest in each case are the blue-coloured ones.

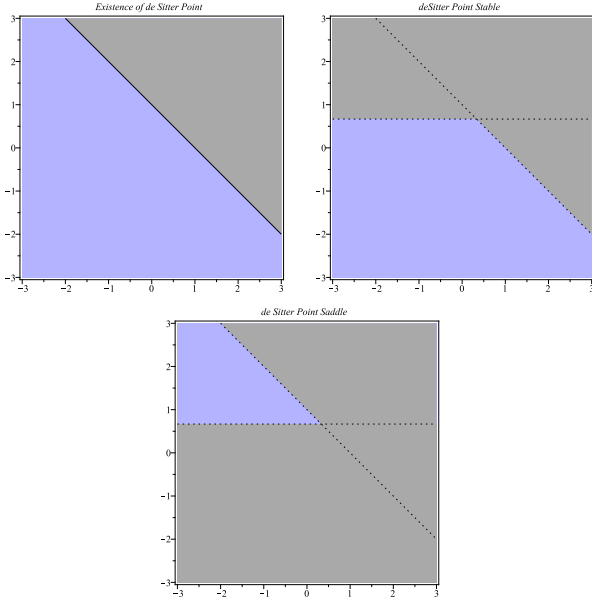


FIG. 12: Existence and stability of the de Sitter critical point in parameters space (ζ_1, ζ_2) for the case IV B. The regions in blue colour are the ones of interest in each case.

which is also dominated by the bulk viscous matter ($x = 1$). The eigenvalues of the Jacobian matrix are:

$$\lambda_1 = -\frac{3(1 - \zeta_1 - \zeta_2)}{2 - 3\zeta_2}, \quad \lambda_2 = -\frac{3}{(2 - \zeta_1 - \zeta_2)^2}.$$

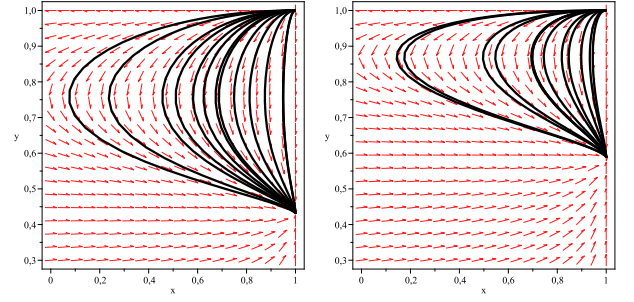


FIG. 13: Phase portraits (x, y) for the case IV B, for different choices of the free parameters ζ_1 , and ζ_2 ($\omega = 0$). We have taken the best estimated values of $\zeta_1 = \tilde{\zeta}_1/3$, and $\zeta_2 = 6\tilde{\zeta}_2$ ($\tilde{\zeta}_0 \neq 0$) from table I: $\tilde{\zeta}_1 = -0.96$, $\tilde{\zeta}_2 = 0$ – left-hand panel, and $\tilde{\zeta}_1 = 0$, $\tilde{\zeta}_2 = 0.05$ – right-hand panel. In both cases the orbits are repelled from the viscous matter-dominated solution and approach to the de Sitter future attractor.

The existence and stability of the de Sitter point in (ζ_1, ζ_2) -parameter space is depicted in Fig.12.

A distinguishing feature of the bulk viscous fluid scenario is the lack of equilibrium points in the phase space that could be correlated with conventional matter-dominance. A phase of conventional matter-dominance is required for the formation of the observed amount of cosmic structure. This is one of the most unwanted features of the viscous fluid scenario and, as shown, this conclusion is irrespective of whether we consider $\omega \neq 0$, or, $\omega = 0$. In the next section we shall see that this conclusion is robust enough and it holds true even if add a radiation component to the model.

V. MODEL OF BULK VISCOUS MATTER WITH CONVENTIONAL MATTER AND RADIATION

In this section we shall investigate a more physically involved scenario with bulk viscous matter, with the bulk viscosity coefficient given by (5). Here, besides a pressureless ($\omega = 0$) viscous matter component, we shall include conventional (non-relativistic or dust) matter, and also radiation. The cosmological equations are the following:

$$\begin{aligned} 3H^2 &= \rho_r + \rho_m + \rho_v, \\ 6\dot{H} + 6H^2 &= -2\rho_r - \rho_m - \rho_v + 9\zeta H, \\ \dot{\rho}_r + 4H\rho_r &= 0, \quad \dot{\rho}_m + 3H\rho_m = 0, \\ \dot{\rho}_v + 3H\rho_v - 9\zeta H^2 &= 0, \end{aligned} \quad (31)$$

where ρ_r is the energy density of the radiation component and, as before, ρ_m and ρ_v stand for the energy densities of non-relativistic (pressureless) matter and of bulk viscous (also pressureless) component, respectively.

In order to transform the above system of equations into a system of autonomous ODE we introduce the following variables of the phase space (the two first variables x , and y , coincide with the former definitions and we add a new variable z):

$$x = \Omega_v, y = \frac{1}{\zeta_0/H + 1}, z \equiv \Omega_r = \frac{\rho_r}{3H^2}. \quad (32)$$

The following autonomous system of ODE is obtained,

$$\begin{aligned} x' &= 3(1-x)\frac{\zeta}{H} + xz, \\ y' &= \frac{y(y-1)}{2} \left(z + 3 - 3\frac{\zeta}{H} \right), \\ z' &= z \left(z - 1 - 3\frac{\zeta}{H} \right), \end{aligned} \quad (33)$$

where

$$\frac{\zeta}{H} = \frac{2 + (2\zeta_1 - (z+1)\zeta_2 - 2)y}{(2 - 3\zeta_2)y}. \quad (34)$$

The Friedmann constraint can be written as, $\Omega_m = 1 - x - z$, while the bounded 3D phase space is given by

$$\Psi = \{(x, y, z) | 0 \leq x \leq 1, 0 < y \leq 1, 0 \leq z \leq 1\}. \quad (35)$$

For the deceleration parameter $q = -1 - H'/H$, and the effective EoS parameter $\omega_{eff} = -1 - 2H'/3H$, one obtains,

$$q = \frac{1 + z - 3\zeta/H}{2}, \quad \omega_{eff} = \frac{z}{3} - \frac{\zeta}{H},$$

respectively.

A. Critical Points and Their Properties

Three equilibrium points are found in the phase space Ψ (35). A summary of these points, $P_i : (x_i, y_i, z_i)$, together with their main features is given below.

1. Bulk viscous matter/radiation-scaling

$$\begin{aligned} P_{r/v} &: (3(\zeta_2 - \zeta_1), 1, 1 - 3(\zeta_2 - \zeta_1)) \\ \Rightarrow \frac{\Omega_r}{\Omega_v} &= \frac{1 - 3(\zeta_2 - \zeta_1)}{3(\zeta_2 - \zeta_1)}. \end{aligned}$$

This solutions exists if

$$0 \leq 3(\zeta_2 - \zeta_1) \leq 1,$$

i. e., when $\zeta_1 \leq \zeta_2 \leq \zeta_1 + 1/3$. In this case the cosmic expansion is decelerating, $q = 1$, while

$\omega_{eff} = 1/3$. The eigenvalues of the Jacobian matrix for this point are,

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = \frac{2[1 - 3(\zeta_2 - \zeta_1)]}{2 - 3\zeta_2},$$

so that it is a unstable critical point in Ψ if $\zeta_2 < 2/3$, and a saddle point otherwise.

2. Bulk viscous matter-dominance, $P_v : (1, 1, 0) \Rightarrow \Omega_v = 1$. This point is characterized by

$$q = \frac{1 - 3\zeta_1}{2 - 3\zeta_2}, \quad \omega_{eff} = \frac{\zeta_2 - 2\zeta_1}{2 - 3\zeta_2},$$

and by the following eigenvalues of its linearization matrix:

$$\begin{aligned} \lambda_1 &= -\frac{2[1 - 3(\zeta_2 - \zeta_1)]}{2 - 3\zeta_2}, \quad \lambda_2 = \frac{3(1 - \zeta_1 - \zeta_2)}{2 - 3\zeta_2}, \\ \lambda_3 &= \frac{3(\zeta_2 - 2\zeta_1)}{2 - 3\zeta_2}. \end{aligned}$$

Stability properties of this equilibrium point in the space of parameters (ζ_1, ζ_2) , are shown in fig.14.

3. de Sitter (also bulk viscous matter-dominated, $\Omega_v = 1$) solution,

$$P_{dS} : \left(1, \frac{1}{2 - \zeta_1 - \zeta_2}, 0 \right) \Rightarrow H = \frac{\zeta_0}{1 - \zeta_1 - \zeta_2}.$$

For this solution $q = \omega_{eff} = -1$. Since the eigenvalues of the Jacobian matrix are,

$$\lambda_1 = -4, \lambda_2 = -3, \lambda_3 = -\frac{3(1 - \zeta_1 - \zeta_2)}{2 - 3\zeta_2},$$

then, whenever it exists, the de Sitter solution is a stable attractor in, Ψ , if $\zeta_2 < 2/3$. Otherwise ($\zeta_2 > 2/3$), it is a saddle point instead (see Fig.14).

A distinctive feature of this model is the absence of critical points associated with conventional dark matter and/or radiation domination. There is only one equilibrium point ($P_{r/v}$) where the radiation and the bulk viscous fluid scale in a constant fraction during the expansion.

As it is suggested by the results of the former section IV, if consider a different EoS parameter $\omega \neq 0$ – say, $\omega = 1/3$, – for the bulk viscous matter, then an additional conventional matter/bulk viscous fluid-scaling critical point $P_{m/v}$ would arise. Hence, in general, there can be found only critical points where the different components of the conventional matter (including radiation) scale with the viscous matter, but, in no case, conventional matter-dominance can be correlated with equilibrium points in the phase space. This result can rule out the bulk viscous matter-dominated models as acceptable models for the description of the cosmological dynamics of our Universe.

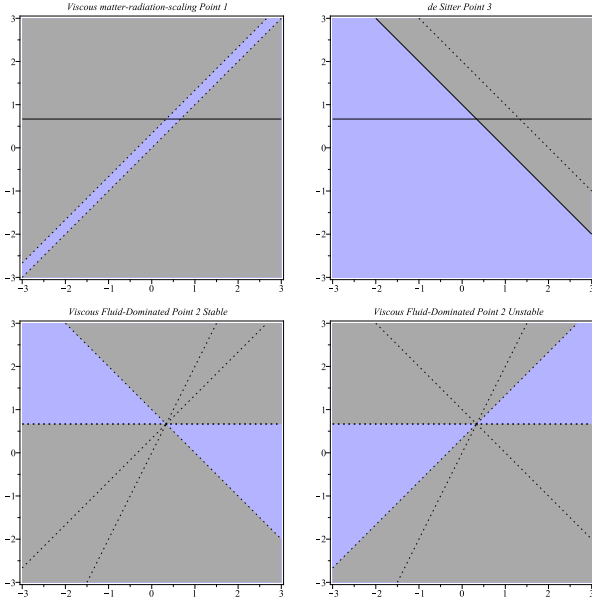


FIG. 14: Parameters space (ζ_1, ζ_2) for the case studied in section V. In the upper left-hand panel we show the region in the space of parameters where the bulk viscous matter/radiation-scaling solution exists (blue strip). Above the horizontal solid line at, $\zeta_2 = 2/3$, this solution represents a saddle point. Below this line it is a past attractor instead. In the upper right-hand panel the blue region in (ζ_1, ζ_2) is the one where the de Sitter solution exists. Above the horizontal solid line marking $\zeta_2 = 2/3$, the de Sitter solution is a saddle point. Below $\zeta_2 = 2/3$ it is a stable attractor. In the lower panels the regions in parameter space where the bulk viscous matter-dominated solution is either a stable attractor (left-hand panel), or a unstable point (right-hand panel), are coloured in blue.

VI. DISCUSSION

In this section we shall make use of the results of a combined investigation of type Ia supernovae test (section III A), and of the dynamical systems study (sections IV and V), to check the viability of the bulk viscous matter model to explain the presently accepted cosmological paradigm. But, before going into the details of the discussion, just a few comments on the role of the study of the equivalent phase space instead of the original set of cosmological equations.

When we replace the original field variables, say H , ρ_m , ρ_v , etc., by the phase space variables x , y , etc., we have to keep in mind that, at the same time, we trade the original set of non-linear, second order differential equations in respect to the cosmological time t (cosmological field equations), by a set of first order ordinary differential equations with respect to the variable $\tau = \ln a$,

$$x' = f(x, y, \dots), \quad y' = g(x, y, \dots), \dots$$

The most important feature of the latter system of ODE is that the differential equations – alternatively, the func-

tions $f(x, y, \dots)$, $g(x, y, \dots)$, etc. – do not depend explicitly on the parameter τ . We call the system autonomous. In other words, we are trading the study of the cosmological dynamics of $H = H(t)$, $\rho_v = \rho_v(t)$, etc., by the study of the flux in τ -parameter of the equivalent autonomous system of ODE. Note that, if consider eternal expansion, $0 < a < \infty$, then t and τ -chronologies are isomorphic;

$$t \in]t_0, \infty[\Rightarrow \tau \in]-\infty, \infty[.$$

The critical (also, equilibrium or fixed) points of this system, $P_i : (x_i, y_i, \dots)$, i. e., the roots of the system of algebraic equations

$$f(x, y, \dots) = 0, \quad g(x, y, \dots) = 0, \dots,$$

correspond to (classes of) solutions of the original system of cosmological equations. If a given equilibrium point $P_a : (x_a, y_a, \dots)$ is a (global) stable attractor, i. e., small linear perturbations around P_a , $x \rightarrow x_a + \xi(\tau)$, $y \rightarrow y_a + \zeta(\tau)$, etc., decay with τ ,⁵ then, independent on the initial conditions chosen, $x(\tau_0) = x_0$, $y(\tau_0) = y_0, \dots$, every orbit in the phase space will approach P_a into the future ($\{\tau : \tau > \tau_0\}$), i. e., the stable (global) attractor is the end point of any orbit in Ψ . On the contrary, if a given critical point, $P_s : (x_s, y_s, \dots)$, were unstable, i. e., small perturbations around, P_s , uncontrollably grow up with τ , then this point were a past attractor or, also, the source point of any orbit in the phase space. For a third class of critical points; the so called "saddle points", depending on the initial conditions chosen, orbits in Ψ can approach to this point, spend some time around it, and then be repelled from it to finally approach to the stable attractor (if it exists).⁶

Now suppose we have a typical phase portrait, composed of a source critical point P_s , a saddle point P_* , and a stable (global) attractor P_a . Each one of these points corresponds to given solutions of the original cosmological equations, $H = H_s(z)$, $H = H_*(z)$, and $H = H_a(z)$, respectively. In the above expressions z is the redshift, which is related with the parameter τ through

$$\tau = -\ln(z + 1), \quad z = z(t) = \frac{1 - a(t)}{a(t)}.$$

⁵ This occurs if the eigenvalues of the Jacobian matrix

$$\begin{pmatrix} f_x & f_y & \dots \\ g_x & g_y & \dots \\ \vdots & \vdots & \dots \end{pmatrix},$$

all have negative real parts.

⁶ For purposes of space, our discussion here is oversimplified, since, in general, critical points can be of many types, for instance, spiral, etc. Besides, there can be found also (un)stable manifolds such as cycles, etc. To worsen things there can coexist several local attractors, saddle points, etc., so that, in general, a given orbit in the phase space can approach to several saddle points before they end up at a given local attractor.

A also typical orbit in the phase space will start at P_s for $\tau = -\infty$, then will approach to P_* , and after a finite (perhaps sufficiently long) $\Delta\tau$, will be repelled by P_* to finally be attracted towards P_a . The parallel history in terms of the equivalent cosmological dynamics will be the following. The expansion starts with a Hubble parameter dynamics $H = H_s(z)$, then, as the Universe expands, the cosmic history enters a transient period characterized by the dynamics dictated by $H = H_*(z)$. After a perhaps long yet finite period Δz , the cosmic expansion will abandon the latter phase, to enter into a stage which dynamics obeys $H = H_a(z)$ lasting for ever.

To illustrate the above discussion with a concrete cosmological model, take as an example the so called "lambda-cold-dark-matter" (Λ -CDM) model. The cosmological equations for this model in a flat FRW space-time are,

$$\begin{aligned} 3H^2 &= \rho_m + \Lambda, \\ 2\dot{H} &= -\rho_m, \quad \dot{\rho}_m + 3H\rho_m = 0, \end{aligned}$$

where ρ_m is the energy density of the CDM and Λ is the cosmological constant. It is convenient to introduce the variable $x \equiv \Lambda/3H^2$ of the phase segment $\{x|0 \leq x \leq 1\}$. The Friedmann constraint can be written as $\Omega_m = 1 - x$. The autonomous ODE obtained in this case is the following: $x' = 3x(1 - x)$. There are two critical points of this ODE: i) the matter dominated solution $x = 0$ ($\Omega_m = 1$), which happens to be a source point (past attractor), and ii) the de Sitter solution $x = 1$ ($H = \sqrt{\Lambda/3}$), which is the stable (future) attractor. Hence, since a typical trajectory in the phase segment starts at $x = 0$ and end ups at $x = 1$, then in this model the cosmic history starts in a matter dominated period (essential for the formation of structure) and end ups in an inflationary stage lasting for ever into the future.

If we improve this model by adding a radiation matter component, there would be three critical points in the (now 2D) phase space: i) a radiation-dominated phase (unstable critical point), followed by ii) a transient matter-dominated stage (saddle point), and iii) a de Sitter point (stable attractor). In this case the cosmic history starts at a radiation-dominated stage, then enters a transient period of matter dominance (essential for the formation of cosmic structure), to finally approach to the stable attractor (the de Sitter phase) which will last for ever into the future. This is, precisely, the behavior one expects from a model designed to recreate the presently adopted cosmological paradigm. One has to care only about giving appropriate initial conditions $x_0 = x(\tau_0)$, etc., so as to get enough formation of structure, i. e., to ensure that the corresponding orbit in the phase space will spend enough time in the neighbourhood of the saddle critical point associated with the matter-dominated solution.

We want to underline that, the fact that a given exact solution of the cosmological equations can not be associated with a critical point in the equivalent phase space,

does not mean at all that the above solution does not exist. In fact it might exist, but it should not be a stable solution, so it should not be useful in a cosmological setting. To illustrate our point take as an example, again, the Λ -CDM model. There is a large set of known (classes of) solutions to the cosmological equations of the model, however, only the radiation-dominated, matter-dominated, and the de Sitter solutions, are of importance in a cosmological context which is compatible with the accepted cosmological paradigm. The latter is characterized by the following stages: i) early time inflationary period, followed by ii) a radiation-dominated, and iii) a matter-dominated phases, both associated with a stage of decelerated expansion, and iv) a present period of accelerated expansion which might last forever.

What happens in the model of interest in this paper? Here, without loss of generality, we shall analyse only the model studied in section V where the cosmological dynamics is fuelled by radiation, conventional pressureless matter, and bulk viscous (also pressureless) matter. There are found three critical points ($\zeta_0 \neq 0$) of the equivalent autonomous system of ODE corresponding to this model: i) bulk viscous matter/radiation-scaling solution $P_{v/r}$, ii) bulk viscous matter dominance P_v , and iii) de Sitter (also bulk viscous matter dominated) solution P_{dS} . If set $\zeta_0 = 0$ only the bulk viscous matter/radiation-scaling solution (point $P_{v/r}$), and the bulk viscous matter-dominated point P_v , survive. In what follows we shall focus in the physically more interesting case where $\zeta_0 \neq 0$ which shows a richer phase space dynamics. Regions in the space of parameters (ζ_1, ζ_2) – case with $\zeta_0 \neq 0$ – where the above critical points exist as well as their stability properties are shown in the figure 14. It is seen that while the point $P_{v/r}$ exists in a very narrow strip in the phase space, the bulk viscous matter-dominated point P_v always exists.

There are four regions of interest in the parameters space (ζ_1, ζ_2):

- R1: region where the three points co-exist together. In this region the viscous matter/radiation-scaling solution ($P_{v/r}$) is the past attractor, the bulk viscous matter-dominated phase (P_v) is a saddle point, while the de Sitter solution (P_{dS}) is the stable attractor.
- R2: region where only $P_{v/r}$ and P_v co-exist. In this case the bulk viscous matter-dominated solution (point P_v) is the past attractor, while, the viscous matter/radiation-scaling solution is a saddle point in the phase space. Not of cosmological interest since there is not critical point which can be associated with a present stage of accelerated expansion.
- R3: region where only the critical point P_v – corresponding to viscous matter dominance – and the de Sitter solution (point P_{dS}) co-exist. In this case, either i) P_v is the past attractor and P_{dS} is the stable

attractor, or ii) P_v is a saddle point while P_{dS} continues being the future attractor (no past attractor exists), or iii) P_{dS} is a saddle point while P_v is the stable attractor (no past attractor exist). The latter case is not of interest for cosmology since there is not any critical point which could be associated with decelerated expansion.

- R4: region where the only existing critical point is P_v – bulk viscous matter dominance. In this region P_v is either a saddle or an unstable critical point. Not of interest for cosmology since there is not any critical point corresponding to the present stage of accelerated expansion.

As seen, only regions R1, R3 i), and R3 ii) could be of cosmological interest. If substitute ζ_1 and ζ_2 by their best estimated values in table I, in both cases the region of parameters space R1 above is single out. In this case $P_{v/r}$ is the past attractor, P_v is a saddle point (both $P_{v/r}$ and P_v are associated with decelerated expansion), while P_{dS} is the stable attractor. As mentioned, in R1 there is an equilibrium point (point $P_{v/r}$) where the bulk viscous fluid scales in a constant fraction with radiation. In this case the effective fluid behaves like radiation, but one which is partly bulk viscous. As already mentioned in the introduction, in a cosmological setting the bulk viscosity may arise when the cosmic fluid expands (or contracts) too fast so that the system does not have enough time to restore its local thermodynamic equilibrium and, then, it arises an effective pressure restoring the system to its thermal equilibrium. When the fluid reaches again the thermal equilibrium the bulk viscous pressure vanishes [5, 6]. This would mean that the stage of the expansion associated with $P_{v/r}$ may last for just a brief period of time while the Universe was out of thermodynamic equilibrium, perhaps not enough to produce the right peace of the growth of the fluctuations. If there is a very brief period of radiation domination, then, as long as the fluctuations re-enter the horizon these will not be sufficiently damped, and correspondingly, an unacceptable large value of the dispersion of the density contrast at the scale $8 h^{-1} \text{Mpc}$ (σ_8) might be obtained (see a similar discussion but in the opposite direction in Ref.[13]). Besides, in either case although in the bulk viscous matter-dominated stage the effective fluid mimics dust, viscosity may affect the formation of structure in a way that can be observationally tested.

As seen from the above analysis the most serious objection against bulk viscous matter scenarios is the absence of conventional matter and radiation dominated eras. Such a behavior is in marked contradiction with the big bang paradigm according to which, back enough into the past when the temperature of the Universe was larger than $10^4 K$, the dynamics of the cosmic evolution was driven by a relativistic mixture in the form of radiation (radiation dominated stage). As the Universe expanded and cooled down to temperatures below

$10^4 K$ (and up to $3 \times 10^3 K$), the density of radiation diluted and the cosmic evolution entered in a stage of (non-relativistic) matter dominance. During this phase radiation decoupled from baryons to form the cosmic microwave background and, what is more important, atoms and the derived cosmic structure we see (galaxies, clusters of galaxies, etc.) were formed.

Here we have shown that the absence of such conventional matter and radiation-dominated phases is irrespective of the region in the space of parameters ζ_0 , ζ_1 , ζ_2 , and ω , so that our conclusion is robust enough and the bulk viscous matter-dominated scenario should be ruled out, at least for the parametrization considered in this paper.

VII. CONCLUSION

In this paper we have applied the dynamical systems tools, in conjunction with the SNe Ia data testing, to judge about the possibility that cosmological bulk viscous matter can stand for an alternative to dark energy. We chose a formerly used parametrization of bulk viscosity [7, 8] with the addition of a term measuring the influence of the acceleration of expansion:

$$\zeta = \zeta_0 + \zeta_1 H + \zeta_2 \left(\frac{\ddot{a}}{\dot{a}} \right).$$

The study of the asymptotic properties of the model in the equivalent phase space shows that there are not critical points that could be associated with either conventional radiation or matter dominance. This result is independent of the values taken by the free parameters of the model. In consequence, the bulk viscous matter-dominated model is not able to accommodate the presently accepted cosmological paradigm. This argument alone can be considered as a serious objection against cosmological models of bulk viscosity. Notwithstanding, we recommend that other possible parametrizations of bulk viscosity should be considered before concluding to rule out the models.

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